Factors that predict better synchronizability on complex networks

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While shorter characteristic path length has in general been believed to enhance synchronizability of a coupled oscillator system on a complex network, the suppressing tendency of the heterogeneity of the degree distribution, even for shorter characteristic path length, has also been reported. To see this, we investigate the effects of various factors such as the degree, characteristic path length, heterogeneity, and betweenness centrality on synchronization, and find a consistent trend between the synchronization and the betweenness centrality. The betweenness centrality is thus proposed as a good indicator for synchronizability.

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In recent years, diverse systems in nature have been observed to exhibit the characteristics of complex networks, drawing much attention to complex network systems [1, 2, 3]. Previous studies have mostly been focused on structural properties of the networks rather than dynamical ones, even though dynamical properties are also very important for understanding the systems on complex networks. In a few studies [3, 4, 5], on the other hand, the dynamical system of coupled oscillators has been considered on complex networks and collective synchronization displayed by the system has been investigated. There it has been found that shorter characteristic path length tends to enhance synchronization. In contrast to this, a recent paper [6], investigating the effects of heterogeneity of the degree distribution on collective synchronization, reported that synchronizability is suppressed as the degree distribution becomes more heterogeneous, even for shorter characteristic path length. These different results then raise a question as to synchronization on complex networks: What is the most important ingredient for better synchronizability?

As an attempt to give an answer to this, we in this paper consider a system of coupled limit-cycle oscillators on the Watts-Strogatz (WS) small-world network [3], and investigate collective synchronization of the system. We pay particular attention to how the synchronization is affected by various factors such as the maximum degree, characteristic path length, heterogeneity of the degree distribution, and betweenness centrality. Here the collective synchronization is explored via the eigenvalues of the coupling matrix, which describes the stability of the fully synchronized state [5, 7].

The WS small-world network is constructed in the following way [3]: We first consider a one-dimensional regular network of N nodes under periodic boundary conditions, with only local connections of range r between the nodes. Next, each local link is visited once, and with the rewiring probability p it is removed and reconnected to a randomly chosen node. At each node of the small-world network built as above, an oscillator is placed; a link connecting two nodes represents coupling between the two oscillators at those two nodes. We now investigate the synchronization of the coupled oscillators on the small-world network with given r and p. Describing the state of the ith oscillator (i.e., the one at node i) by x_i, we begin with the set of equations of motion governing the dynamics of the N coupled oscillators:

$$\dot{x}_i = F(x_i) + K \sum_{j=1}^{N} M_{ij} G(x_j), \tag{1}$$

where $\dot{x}_i = F(x_i)$ governs the dynamics of individual oscillators (i.e., with coupling strength $K = 0$) and $G(x_j)$ makes the output function. The $N \times N$ coupling matrix M_{ij} is given by

$$M_{ij} = \begin{cases} k_i & \text{for } i = j \\ -1 & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

which lacks the translational symmetry due to the presence of shortcuts on the WS small-world network. In the case of a locally-coupled (hypercubic) network with the coordination number $z\,(= 2D)$, the coupling matrix M_{ij} has the value z on the diagonal and -1 on the z off-diagonals adjacent to the diagonal.

The eigenvalues of the coupling matrix have been widely used to determine the linear stability of the fully synchronized state ($x_1 = x_2 = \cdots = x_N$) [5, 7]. Whereas the smallest eigenvalue, denoted by λ_0, is always zero, the ratio of the maximum eigenvalue $\lambda_{\max}$ to the smallest *non-vanishing* one $\lambda_{\min}$ may be used as a measure for the stability of the synchronized state, with larger values of the ratio $\lambda_{\max}/\lambda_{\min}$ corresponding to poor synchronizability [5, 7]. For a general D-dimensional hypercubic network of linear size L, the eigenvalues are given by

$$\lambda_{\{m_\alpha\}} = 4 \sum_{\alpha=1}^{D} \sin^2 \frac{\pi m_\alpha}{L}, \tag{3}$$

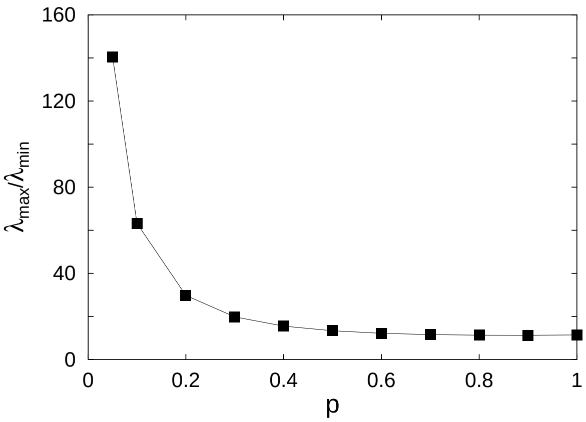

FIG. 1: Behavior of the ratio of the maximum eigenvalue $\lambda_{\max}$ to the smallest non-vanishing eigenvalue $\lambda_{\min}$ with the rewiring probability p. As p is raised, the ratio is shown to reduce, yielding better synchronization.

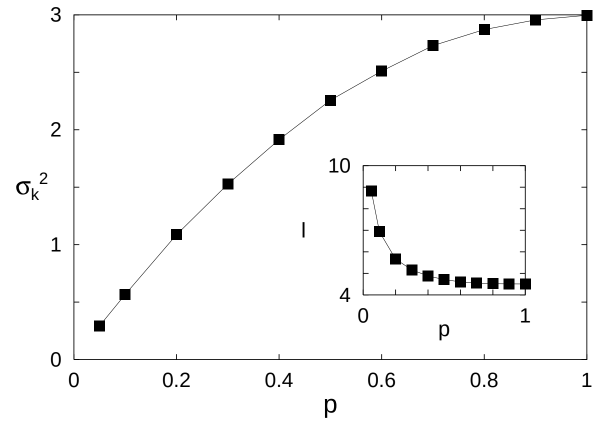

FIG. 2: Behavior of the variance ${\sigma_k}^2$ of the degree distribution with the rewiring probability p. Inset: characteristic path length ℓ vs. the rewiring probability p. Here and in subsequent figures lines are merely guides to eyes.

where $m_\alpha = 0, 1, \cdots, L{-}1$ and $L^D \equiv N$. It is then obvious that the eigenvalue ratio behaves as $\lambda_{\max}/\lambda_{\min} \sim N^{2/D}$ and grows large in the thermodynamic limit ($N \to \infty$). It is thus concluded that the fully synchronized state is not stable in any D-dimensional regular network.

The eigenvalue ratio for the WS network of nodes $N = 2000$ and range $r = 3$ is obtained numerically and its behavior with the rewiring probability is exhibited in Fig. 1, where the average has been taken over 100 different network realizations. The network size N has been varied from 100 to 2000, only to give no qualitative difference. As the rewiring probability p is increased, the ratio $\lambda_{\max}/\lambda_{\min}$ is observed to decrease, which implies enhancement of synchronizability.

To explore how structural properties of the underlying network affect synchronization of the system, we now examine such properties as the characteristic path length, the betweenness centrality, and the variance of the degree distribution, which is a measure of the heterogeneity of the degree distribution. Figure 2 displays the behavior of the variance ${\sigma_k}^2 \equiv \langle N^{-1}\sum_i {k_i}^2\rangle - \langle (N^{-1}\sum_i k_i)^2\rangle$ of the degree distribution for the WS small-world network with the same size $N = 2000$ and range $r = 3$ as that in Fig. 1, depending on the rewiring probability p. As p is increased, the variance ${\sigma_k}^2$ grows, which implies that the degree distribution becomes more broad and heterogeneous and that nodes of larger degrees appear. In the inset of Fig. 2 the behavior of the characteristic path length [8]

$$\ell \equiv \left\langle \frac{1}{N(N-1)} \sum_{i,j} d_{i,j} \right\rangle, \tag{4}$$

where $\langle\cdots\rangle$ denotes the average over different realizations of the network and $d_{i,j}$ the length of the geodesic between i and j, is shown as a function of the rewiring probability p. As expected, the characteristic path length ℓ is observed to decrease as the heterogeneity of the degree distribution (or the rewiring probability) is increased. The results shown in Figs. 1 and 2 imply that the synchronizability on the WS network is improved as the heterogeneity of the degree distribution is increased or as the characteristic path length is decreased, which differs from the behavior observed in scale-free networks [6].

The synchronizability has been shown to be related with the load or betweenness centrality on nodes [6]. The betweenness centrality of node n is defined to be [9, 10, 11, 12]:

$$B_n \equiv \sum_{(i,j)} \frac{g_{inj}}{g_{ij}}, \tag{5}$$

where g_{ij} is the number of geodesic paths between nodes i and j and g_{inj} is the number of paths between i and j passing through node n. The summation is to be performed over all pairs of nodes (i, j) such that $i, j \neq n$ and $i \neq j$.

To get an idea of the betweenness centrality, which measures how many geodesics pass through a given node, we first consider locally-coupled regular networks, for which the average betweenness centrality is given by [11]

$$\bar{B} \equiv \frac{1}{N} \sum_n B_n = (N-1)(\ell-1). \tag{6}$$

Among the N values of B_n's, the maximum value $B^{\max}$ has been shown to be related with synchronizability [6], although this close relation has not been stressed before. For a D-dimensional local regular network, we have $\bar{B} = B^{\max}$ and the characteristic path length $\ell \sim N^{1/D}$, which yields

$$B^{\max} \sim N^{(D+1)/D}, \tag{7}$$

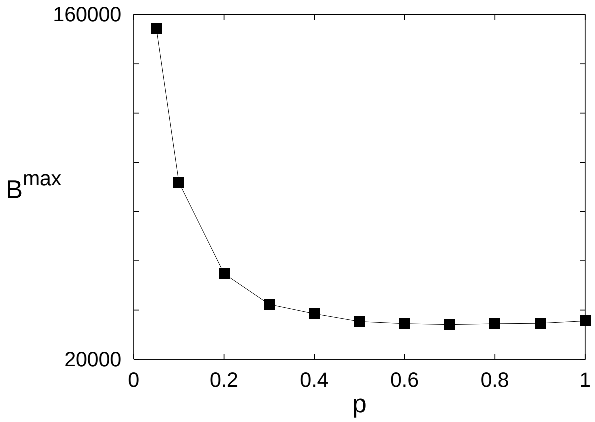

FIG. 3: Behavior of the maximum value $B^{\max}$ of the betweenness centrality with the rewiring probability p. $B^{\max}$ is shown to decrease as p is increased, corresponding to more heterogeneous degree distributions.

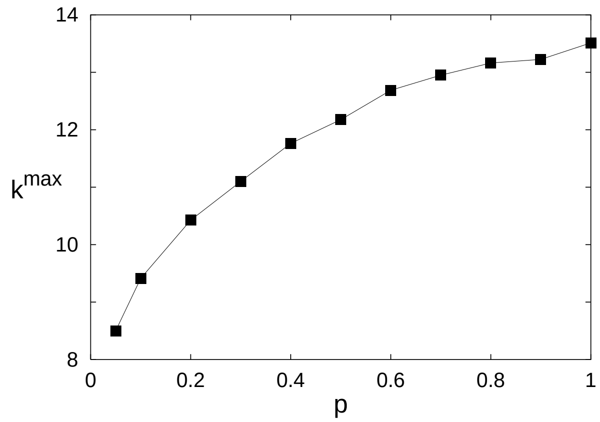

FIG. 4: The maximum degree $k^{\max}$ versus the rewiring probability p on a WS small-world network. It is shown that $k^{\max}$ increases as the degree distribution becomes more heterogeneous.

i.e., $B^{\max} \sim N^2$, $N^{3/2}$, and $N^{4/3}$ for the spatial dimension $D = 1$, 2, and 3, respectively. Thus the maximum value $B^{\max}$ increases algebraically with the size N of the regular network, although the exponent reduces with the spatial dimension D.

Returning to the WS small-world network, we compute the betweenness centrality B_n via a modified version of the breadth-first search algorithm [10]. We then obtain the maximum value $B^{\max}$ at various values of the rewiring probability, and display the result for a network of nodes $N = 2000$ in Fig. 3, where the average has been taken over 100 different network realizations. The number of nodes N has also been varied from $N = 100$ to $N = 2000$, which does not yield any qualitative difference. Figure 3 shows that the maximum load on a node reduces as more shortcuts are introduced. In general, on a usual scale-free network, larger values of $B^{\max}$ correspond to larger values of the degree. For comparison, we also investigate the maximum degree $k^{\max}$ on our small-world network and display in Fig. 4 its behavior with the rewiring probability p. The increase of $k^{\max}$ with p indicates the opposite trend between $B^{\max}$ and $k^{\max}$, unlike the case of a scale-free network. Note also that Fig. 4 together with Fig. 2 implies the increase of the maximum degree with the heterogeneity of the degree distribution.

The results shown in Figs. 1 to 4 lead to the conclusion that synchronization on the WS network is enhanced as the heterogeneity of the degree distribution is increased, as the characteristic path length decreased, as the maximum betweenness centrality decreased, or as the maximum degree increased. Remarkably, the effects of the heterogeneity of the degree distribution as well as those of the characteristic path length (see Fig. 2) differ from the results for other classes of networks [6]. Namely, in the case of the network studied here, larger heterogeneity of the degree distribution or shorter characteristic path length does not improve synchronizability. On the other hand, the effects of the maximum betweenness centrality $B^{\max}$ appear to be consistent with those in Ref. [6]: Synchronizability is always improved as $B^{\max}$ is reduced. Accordingly, the betweenness centrality is proposed as a suitable indicator for predicting synchronizability on complex networks. Regarding the maximum degree $k^{\max}$, Fig. 4 (together with Fig. 1) indicates that synchronizability enhances with $k^{\max}$. This behavior on the WS small-world network is also in contrast with that on a usual scale-free network, where $B^{\max}$ and $k^{\max}$ behave similarly [12], and accordingly, smaller values of $k^{\max}$ are expected to give better synchronizability.

Then why does the maximum value $B^{\max}$ of the betweenness centrality strongly affect synchronizability on networks? An intuitive argument goes as follows: Suppose that two groups of highly linked nodes are connected via a few nodes. Among those a few connecting nodes, the information of the synchronized state passes through the node which has the maximum value of the betweenness centrality, $B^{\max}$. It tends to get overloaded since most paths go through it, which in turn leads to loss of information. Accordingly, synchronizability is expected to become reduced as the value $B^{\max}$ is increased. This argument is consistent with that of Ref. [6]. To check this argument, we have examined synchronizability on the WS network before and after the removal of the node having $B^{\max}$, by means of the eigenvalue ratio $\lambda_{\max}/\lambda_{\min}$. In Fig. 5, the difference $\delta \equiv (\lambda_{\max}/\lambda_{\min})_{\text{after removal}} - (\lambda_{\max}/\lambda_{\min})_{\text{before removal}}$ is plotted against the rewiring probability p. Squares in Fig. 5 represent the difference δ for the removal of the node having $B^{\max}$, and indicate negative values regardless of the rewiring probability p. Those negative values imply that $\lambda_{\max}/\lambda_{\min}$ is reduced after the removal

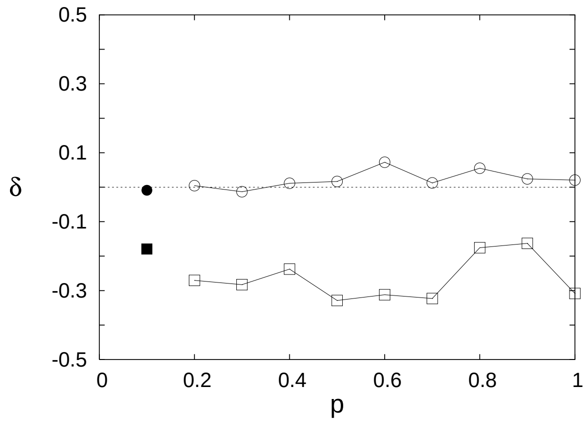

FIG. 5: Behavior of the difference δ of the eigenvalue ratio is displayed with the rewiring probability p. The data (open circles and squares) have been obtained for the network with the number of nodes $N = 2000$ and $r = 3$. Due to strong finite size effects at $p = 0.1$, we use the data (filled circle and square) for large system size of $N = 5000$. See the text for the explanation of the data symbols.

of the node, which in turn implies that synchronizability of the system is enhanced after the removal of the node. In sharp contrast, circles in Fig. 5, which represent the difference δ for the removal of a node other than the one with $B^{\max}$, indicate values near zero. This implies that the eigenvalue ratio remains almost the same when an arbitrary node is removed and accordingly that synchronizability of the system is not much affected by the removal of an arbitrary node. The result displayed in Fig. 5 thus manifests that the node with $B^{\max}$ is closely related with the synchronizability of the system whereas any other node is not substantially related.

In conclusion, better synchronizability for the WS small-world network is induced as the heterogeneity of the degree distribution is increased, as the characteristic path length is decreased, as the maximum betweenness centrality is reduced, or as the maximum degree is raised. We have found that the effects of the characteristic path length and of the heterogeneity on synchronization in the WS small-world network are different from those in the networks considered in Ref. [6]. These differences seem to be related with the presence of hub structures in the network, which is under further investigation. Our result implies that shorter characteristic path length or larger heterogeneity does not always enhance the synchronizability of the coupled system on a network. On the other hand, it has been observed that synchronization is always enhanced as the betweenness centrality, measuring the load on a node, is reduced, which is consistent with the recent result for various networks [6]. We have also numerically investigated the effects of the node of the maximum betweenness centrality $B^{\max}$ on synchronizability, and found that the node of $B^{\max}$ is highly related with the synchronizability of the system, which supports the main conclusion of this paper. It is thus concluded that among the important factors for better synchronization on complex networks is a small value of the maximum betweenness centrality, rather than short characteristic path length or large heterogeneity of the degree distribution.

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